TOWARDS DOMAIN DECOMPOSITION FOR NONLOCAL PROBLEMS

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Abstract. In this paper we present the first results on substructuring methods for nonlocal operators, specifically, an instance of the nonlocal p-Laplace operator. We present a nonlocal variational formulation of this operator, proving a nonlocal Poincaré inequality and upper bound to establish a spectral equivalence. We then introduce a nonlocal two-domain variational formulation utilizing nonlocal transmission conditions, and prove equivalence with the single-domain formulation. A nonlocal Schur complement is introduced. We establish condition number bounds for the nonlocal stiffness and Schur complement matrices. Supporting numerical experiments demonstrating the conditioning of the nonlocal single- and two-domain problems are presented.

Key words. Domain decomposition, nonlocal substructuring, nonlocal operators, nonlocal Poincaré inequality, p-Laplacian, peridynamics, nonlocal Schur complement, condition number.

AMS subject classifications. 34B10, 65N55, 45A05, 35A15.

1. Introduction.

Domain decomposition methods where the subdomains do not overlap are called *substructuring methods*, reflecting their origins and long use within the structural analysis community [29]. These methods solve for unknowns only along the interface between subdomains, thus decoupling these domains from each other and allowing each subdomain to then be solved independently. One may solve for the primal field variable on the interface, generating a Dirichlet boundary value problem on each subdomain (these are Schur complement methods, see [39] and references cited therein), or solve for the dual field variable on the interface, generating a Neumann boundary value problem on each subdomain (these are dual Schur complement methods, see [21, 22, 20, 31]). Hybrid dual-primal methods have also been developed [26].

As domain decomposition methods are frequently employed on massively parallel computers, only *scalable* methods are of interest, meaning that the condition number of the interface problem does not grow (or, only grows weakly) with the number of subdomains. Scalable or weakly scalable methods are generated by application of an appropriate preconditioner to the interface problem. This preconditioner requires the solution of a coarse problem to propagate error globally; see any of the references [6, 7, 8, 9, 19, 28, 27, 15]. For a general overview of domain decomposition, the reader is directed to the excellent texts [39, 30, 40].

All of the methods referenced above have in common that they are domain decomposition approaches for *local* problems. In this paper, we propose and study a domain decomposition method for the *nonlocal* Dirichlet boundary value problem

$$\mathcal{L}(\mathbf{u}) = \mathbf{b}(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \tag{1.1}$$

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where

$$\mathcal{L}(\mathbf{u}) := \int_{\Omega \cup \mathcal{B}\Omega} \mathbf{C}(\mathbf{x}, \mathbf{x}')\,(\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}))\,d\mathbf{x}', \tag{1.2}$$

Ω is a bounded domain, $\mathcal{B}\Omega$ is given in (2.1), $\mathbf{b}$ is given, and $\mathbf{u}(\mathbf{x}) \in \mathbb{R}^d$ is prescribed for $\mathbf{x} \in \mathbb{R}^d \backslash \Omega$. We prescribe the value of $\mathbf{u}(\mathbf{x})$ outside Ω and not just on the boundary of Ω, owing to the nonlocal nature of the problem.

Nonlocal models are useful where classical (local) models cease to be predictive. Examples include porous media flow [13, 14, 38], turbulence [5], fracture of solids, stress fields at dislocation cores and cracks tips, singularities present at the point of application of concentrated loads (forces, couples, heat, etc.), failure in the prediction of short wavelength behavior of elastic waves, microscale heat transfer, and fluid flow in microscale channels [18]. These are also cases where microscale fields are nonsmooth. Consequently, nonlocal models are also useful for multiscale modeling. Recent examples of nonlocal multiscale modeling include the upscaling of molecular dynamics to nonlocal continuum mechanics [32], and development of a rigorous multiscale method for the analysis of fiber-reinforced composites capable of resolving dynamics at structural length scales as well as the length scales of the reinforcing fibers [1]. Progress towards a nonlocal calculus is reported in [25]. Development and analysis of a nonlocal diffusion equation is reported in [2, 3, 4]. Theoretical developments for general class of integro-differential equation related to the fractional Laplacian are presented in [10, 11, 37]. We discuss in §2 some specific contexts where the nonlocal operator $\mathcal{L}$ appears, and the assumptions placed upon $\mathcal{L}$ by those interpretations.

To the best of authors' knowledge, this paper represents the first work on domain decomposition methods for nonlocal models. Our aim is to generalize iterative substructuring methods to a nonlocal setting and characterize the impact of nonlocality upon the scalability of these methods. To begin our analysis, we first develop a weak form for (1.1) in §3. The main theoretical construction for conditioning is in §4. We establish spectral equivalences to bound the condition numbers of the stiffness and Schur complement matrices. For that, we prove a nonlocal Poincaré inequality for the lower bound and a dimension dependent estimate for the upper bound. We also reveal a striking difference between the local and nonlocal problems. Namely, *the condition number of the discrete nonlocal operator is mesh size independent.* Moreover, unlike the local case, we show that the condition number varies with the spatial dimension. In §5 we formulate two-domain strong and weak forms of (1.1), giving particular attention to the nonlocal transmission conditions. We then prove equivalence of the one-domain and two-domain strong forms, and also equivalence of the one-domain and two-domain weak forms. Another remarkable result is that this equivalence of strong and weak forms can be achieved while allowing $\mathbf{u}$ to be less regular than is required to demonstrate equivalence of two- and one-domain problems for the (local) Laplacian. In §6, we first define an *energy minimizing extension*, a nonlocal analog of harmonic extension in the local case, to study the conditioning of the Schur complement in the nonlocal setting. We discretize our two-domain weak form to arrive at a *nonlocal Schur complement.* We perform numerical studies to validate our theoretical results. Finally in §7, we draw conclusions about conditioning and suggest future research directions for nonlocal domain decomposition methods.

2. Interpretations of the Operator $\mathcal{L}$. The operator $\mathcal{L}$ appears in many different application areas, from evolution equations for species population densities [12] to image processing [24]. We review two specific contexts in which the operator

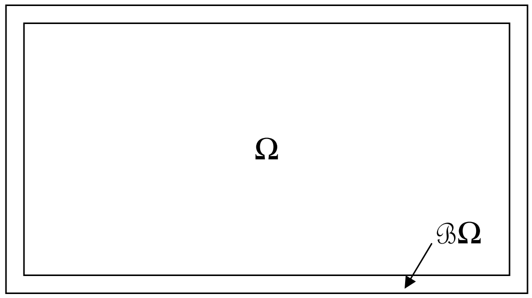


FIG. 2.1. *Typical domain for* (1.1). $\mathbf{u}$ *is prescribed in* $\mathcal{B}\Omega$, *and we solve for* $\mathbf{u}$ *in* Ω.

$\mathcal{L}$ of (1.1) is utilized, paying special attention to associated assumptions these interpretations place upon $\mathbf{C}$ in $\mathcal{L}$. In all cases, we find $\mathbf{C}$ to have local support about $\mathbf{x}$, meaning that we must prescribe Dirichlet boundary conditions only for

$$\mathcal{B}\Omega := \mathrm{supp}(\mathbf{C})\backslash\Omega, \tag{2.1}$$

as depicted in Figure 2.1.

2.1. Nonlocal Diffusion Processes. The equation

$$u_t(\mathbf{x}, t) = \mathcal{L}(u(\mathbf{x}, t)) \tag{2.2}$$

is an instance of a nonlocal p-Laplace equation for $p = 2$, and has been used to model nonlocal diffusion processes, see [4] and the references cited therein. In this setting, $u(\mathbf{x}, t) \in \mathbb{R}$ is the density at the point $\mathbf{x}$ at time t of some material, and $C(\mathbf{x}, \mathbf{x}') = C(\mathbf{x} - \mathbf{x}')$ is the probability distribution of material movement from $\mathbf{x}'$ to $\mathbf{x}$. Then, $\int_{\mathbb{R}^d} C(\mathbf{x}' - \mathbf{x})u(\mathbf{x}', t)d\mathbf{x}'$ is the rate at which material is arriving at $\mathbf{x}$ from all other points in $\mathrm{supp}(C)$, and $-\int_{\mathbb{R}^d} C(\mathbf{x}' - \mathbf{x})u(\mathbf{x}', t)d\mathbf{x}'$ is the rate at which material departs $\mathbf{x}$ for all other points in $\mathrm{supp}(C)$ [23, 4].

In this interpretation of (1.1) the following restrictions are placed upon C in $\mathcal{L}$. It is assumed that $C : \mathbb{R}^d \to \mathbb{R}$ is a nonnegative, radial, continuous function that is strictly positive in a ball of radius δ about $\mathbf{x}$ and zero elsewhere. Additionally, it is assumed that $\int_\Omega C(\boldsymbol{\xi})d\boldsymbol{\xi} < \infty$.

2.2. Nonlocal Solid Mechanics. The equation

$$\mathbf{u}_{tt}(\mathbf{x}, t) = \mathcal{L}(\mathbf{u}(\mathbf{x}, t)) + \mathbf{b}(\mathbf{x}) \tag{2.3}$$

is the linearized peridynamic equation [34, eqn. (56)]. The corresponding time-independent ("peristatic") equilibrium equation is (1.1). Peridynamics is a nonlocal reformulation of continuum mechanics that is oriented toward deformations with discontinuities, see [34, 35, 33] and the references therein. In this context, $\mathbf{u} \in \mathbb{R}^d$ is the displacement field for the body Ω, and $\mathbf{C}(\mathbf{x}, \mathbf{x}')$ is a stiffness tensor.

In this interpretation of (1.1) the following restrictions are placed upon $\mathbf{C}$ in $\mathcal{L}$. It is assumed that $\mathbf{C}$ is bounded and strictly positive definite in the *neighborhood* of $\mathbf{x}$, $\mathcal{H}_\mathbf{x}$, defined by the following with $\delta > 0$:

$$\mathcal{H}_\mathbf{x} := \{\mathbf{x}' \;:\; \|\mathbf{x}' - \mathbf{x}\| \le \delta\}. \tag{2.4}$$

These assumptions are made to ensure material stability [34, pp. 191-194]. It is also assumed that $\mathbf{C} = \mathbf{0}$ for $\|\mathbf{x}' - \mathbf{x}\| > \delta$. If the material is elastic, it follows that $\mathbf{C}(\mathbf{x}, \mathbf{x}')$ is symmetric (e.g., $\mathbf{C}(\mathbf{x}, \mathbf{x}')^T = \mathbf{C}(\mathbf{x}, \mathbf{x}')$). Further, it is assumed that $\mathbf{C}$ is symmetric with respect to its arguments (e.g., $\mathbf{C}(\mathbf{x}, \mathbf{x}') = \mathbf{C}(\mathbf{x}', \mathbf{x})$). This follows from imposing that the integrand of (1.2) must be antisymmetric in its arguments, e.g.,

$$\mathbf{C}(\mathbf{x}, \mathbf{x}') \cdot [\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})] = -\mathbf{C}(\mathbf{x}', \mathbf{x}) \cdot [\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')]$$

in accordance with Newton's third law.

3. A Nonlocal Variational Formulation. Here we present a variational formulation of the nonlocal equation (1.1). For peridynamics, this was presented by Emmerich and Weckner in [17]. An analogous expression also appears in [34, eqn. (75)], as well as [25].

We will utilize the function space

$$V := L^d_{2,0}(\Omega) = \left\{ \mathbf{v} \in L^d_2(\Omega) : \mathbf{v}|_{\mathcal{B}\Omega} = \mathbf{0} \right\}, \tag{3.1}$$

and the inner product

$$(\mathbf{u}, \mathbf{v})_\Omega := \int_\Omega \mathbf{u} \cdot \mathbf{v} \; d\mathbf{x}.$$

The weak formulation of (1.1) is the following: Given $\mathbf{b}(\mathbf{x}) \in V$, find $\mathbf{u}(\mathbf{x}) \in V$ such that

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{b}, \mathbf{v}) \qquad \forall \mathbf{v} \in V, \tag{3.2}$$

where

$$a(\mathbf{u}, \mathbf{v}) := -\int_\Omega \left\{ \int_{\Omega \cup \mathcal{B}\Omega} \mathbf{C}(\mathbf{x}, \mathbf{x}') \cdot [\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})] \; d\mathbf{x}' \right\} \cdot \mathbf{v}(\mathbf{x}) \; d\mathbf{x}. \tag{3.3}$$

It follows from assumptions in §2 that the iterated integral in (3.3) is finite:

$$-\int_\Omega \left\{ \int_{\Omega \cup \mathcal{B}\Omega} \mathbf{C}(\mathbf{x}, \mathbf{x}') \cdot [\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})] \; d\mathbf{x}' \right\} \cdot \mathbf{v}(\mathbf{x}) \; d\mathbf{x} < \infty,$$

and that $\mathbf{C}(\mathbf{x}, \mathbf{x}') \cdot [\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})]$ is anti-symmetric in its arguments. Combining these observations with Fubini's Theorem gives the identity

$$\begin{aligned} -\int_\Omega \left\{ \int_{\Omega \cup \mathcal{B}\Omega} \mathbf{C}(\mathbf{x}, \mathbf{x}') \cdot [\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})] \; d\mathbf{x}' \right\} \cdot \mathbf{v}(\mathbf{x}) \; d\mathbf{x} = \\ \frac{1}{2} \int_\Omega \left\{ \int_{\Omega \cup \mathcal{B}\Omega} \mathbf{C}(\mathbf{x}, \mathbf{x}') \cdot [\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})] \; d\mathbf{x}' \right\} \cdot [\mathbf{v}(\mathbf{x}') - \mathbf{v}(\mathbf{x})] \; d\mathbf{x}. \end{aligned} \tag{3.4}$$

Remark 3.1. *The equivalence in (3.4) can be interpreted as a duality pairing.*
The equivalent expression of $a(\mathbf{u}, \mathbf{u})$ from (3.4),

$$a(\mathbf{u}, \mathbf{u}) = \frac{1}{2} \int_\Omega \int_{\Omega \cup \mathcal{B}\Omega} \mathbf{C}(\mathbf{x}, \mathbf{x}') \cdot [\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})]^2 \; d\mathbf{x}' \; d\mathbf{x}, \tag{3.5}$$

is positive. This immediately establishes coercivity. In the setting of nonlocal solid mechanics, (3.5) has the natural interpretation of the energy stored in deformed elastic

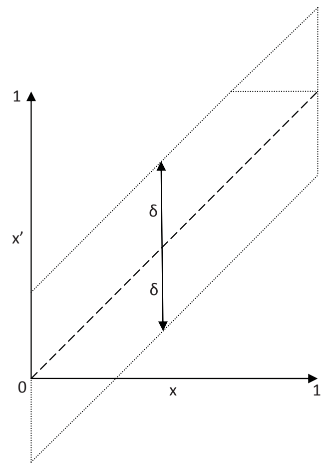


FIG. 3.1. *Domain of integration for 1D problem where $\Omega = [0, 1]$. The integrand is zero outside the region bounded by the dotted line, and is discontinuous across this line. Unlike x, the variable x' is integrated over $\mathcal{B}\Omega = [-\delta, 0] \cup [1, 1 + \delta]$ where u is prescribed as a nonlocal Dirichlet boundary condition.*

solid. Note that the zero energy modes, or rigid body modes, are excluded by enforcing Dirichlet boundary conditions.

In 1D with $\Omega := [0, 1]$, the weak form (3.5) becomes

$$a(u, u) = \frac{1}{2} \int_0^1 \int_{x-\delta}^{x+\delta} C(x, x') \left(u(x') - u(x)\right)^2 dx' dx, \tag{3.6}$$

where the limits of integration have been adjusted to account for the support of $C(x, x')$, which is assumed to vanish if $\|x - x'\| > \delta$. For this problem, the two-dimensional domain of integration is the parallelogram shown in Figure 3.1. For 2D and 3D problems, the domains of integration are four and six dimensional, respectively.

4. Nonlocal Condition Number Estimates. In the local setting, the classical condition number estimates rely on a Poincaré inequality and an inverse inequality for the lower and upper bound, respectively. Similarly to the local case, we develop a nonlocal Poincaré inequality to be used in the lower bound. We prove a Poincaré inequality involving an explicit δ-quantification, which is a more refined result than what is available in the literature [4]. The δ-quantification is an essential feature in the nonlocal setting because the lower bound turns out to be dimension dependent, unlike in the local case. This dimensional dependence is induced by the neighborhood $\mathcal{H}_x$ (see (2.4)), which is d-dimensional in the nonlocal setting but zero-dimensional (a point) in the local setting. Dimension dependence in the Poincaré inequality is captured by δ^m (see §4.1) where the power m exhibits a dimensional dependence (i.e., $m = m(d)$).

For the upper bound, we prove a direct estimate instead of an inverse inequality. Neither the upper bound estimate nor the Poincaré requires discretization. Hence, our estimate is valid in infinite dimensional function spaces, a stronger result than that for the local setting.

Furthermore, as in the classical condition number analysis, the dependence on the kernel function $\mathbf{C}$ is absorbed into the following constants:

$$\overline{\mathbf{C}} := \sup_{\mathbf{x}\in\Omega,\mathbf{x}'\in\Omega\cup\mathcal{B}\Omega} \|\mathbf{C}(\mathbf{x},\mathbf{x}')\|$$
$$\underline{\mathbf{C}} := \inf_{\mathbf{x}\in\Omega,\mathbf{x}'\in\Omega\cup\mathcal{B}\Omega} \|\mathbf{C}(\mathbf{x},\mathbf{x}')\|.$$

Therefore, we reduce the analysis to the *canonical* kernel function $\chi_\delta(\mathbf{x},\mathbf{x}')$ whose only role is the representation of the neighborhood in (2.4) by a characteristic function. Namely,

$$\chi_\delta(\mathbf{x},\mathbf{x}') := \begin{cases} 1, & \|\mathbf{x}-\mathbf{x}'\| \le \delta \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

For the remainder of this paper, will restrict our discussion to scalar problems, e.g., $\mathbf{u}(\mathbf{x}) = u(\mathbf{x})$, $\mathbf{C}(\mathbf{x},\mathbf{x}') = C(\mathbf{x},\mathbf{x}')$, etc.

4.1. Nonlocal Poincaré inequality. We prove a nonlocal Poincaré inequality that is valid for $u(\mathbf{x}) \in L_p(\Omega)$ with $p \ge 1$. We define a generalized bilinear form to support the proof in $L_p(\Omega)$:

$$a^{(p)}(u_g, u) := \int_\Omega \int_{\Omega\cup\mathcal{B}\Omega} \chi_\delta(\mathbf{x},\mathbf{x}')\, |u_g(\mathbf{x}') - u(\mathbf{x})|^p \; d\mathbf{x}' d\mathbf{x}.$$

For the explicit δ-quantification, we utilize inscribing and circumscribing spheres of Ω so that the Poincaré constant λ_{Pncr} remains independent of δ; see Appendix A.

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded, simply connected domain. Assume that $u \in L_p(\Omega)$ and $g \in L_p(\mathcal{B}\Omega)$ with $u_g(\mathbf{x}) := \begin{cases} u(\mathbf{x}), & \mathbf{x}\in\Omega \\ g(\mathbf{x}), & \mathbf{x}\in\mathcal{B}\Omega \end{cases}$ with $p \ge 1$. Then, for sufficiently small δ, there exists $\lambda_{Pncr} = \lambda_{Pncr}(m, p, \Omega) > 0$ with $m \ge 1$ such that*

$$\lambda_{Pncr}\, \delta^m \int_\Omega |u(\mathbf{x})|^p \; d\mathbf{x} \le a^{(p)}(u_g, u) + \delta^m \int_{\mathcal{B}\Omega} |g(\mathbf{x}')|^p \; d\mathbf{x}'. \tag{4.2}$$

Proof. We construct a finite covering for Ω using strips of width $\frac{\delta^m}{2}$ in a similar fashion utilized in [4, Prop. 2.5]:

$$B_0 := \{\mathbf{x} \in \mathcal{B}\Omega : \|(\mathbf{x}, \Omega)\| \le \frac{\delta^m}{2}\}, \tag{4.3}$$
$$B_1 := \{\mathbf{x} \in \Omega : \|(\mathbf{x}, B_0)\| \le \frac{\delta^m}{2}\}, \tag{4.4}$$
$$B_j := \{\mathbf{x} \in \Omega \setminus \bigcup_{k=1}^{j-1} : \|(\mathbf{x}, B_{j-1})\| \le \frac{\delta^m}{2}\}, \quad j = 2, 3, \ldots, l. \tag{4.5}$$

Here, for a fixed δ, the number of strips $l = l(\Omega, m)$. We trivially have the following for $j = 1, \ldots, l$:

$$\int_\Omega \int_{\Omega\cup\mathcal{B}\Omega} \chi_\delta(\mathbf{x},\mathbf{x}')\, |u_g(\mathbf{x}') - u(\mathbf{x})|^p \; d\mathbf{x}' d\mathbf{x} \ge \int_{B_j} \int_{B_{j-1}} \chi_\delta(\mathbf{x},\mathbf{x}')\, |u_g(\mathbf{x}') - u(\mathbf{x})|^p \; d\mathbf{x}' d\mathbf{x}.$$

Note that for $\mathbf{x}' \in B_{j-1}$ and $\mathbf{x} \in B_j$, due to $\delta \le 1$, we have

$$\|\mathbf{x} - \mathbf{x}'\| \le \delta^m \le \delta.$$

We refine the result in [4] by establishing an explicit δ-quantification. For that, we utilize the inscribing and circumscribing spheres of Ω whose radius are denoted by radius r_{in} and r_{out}, respectively. Then, we compare the volume of the strips B_j to the volume of the annuli of width $\frac{\delta^m}{2}$ of the inscribing and circumscribing spheres. For sufficiently small δ, we prove the following in Appendix A:

$$c_{in}\,\delta^m \leq |B_j| \leq c_{out}\,\delta^m, \tag{4.6}$$

where c_{in} and c_{out} depend only on Ω. In fact, $c_{in} \approxeq r_{in}^{d-1}$ and $c_{out} \approxeq r_{out}^{d-1}$.

From (4.1) we have:

$$\chi_\delta(\mathbf{x},\mathbf{x}') = 1, \quad \mathbf{x}' \in B_{j-1},\ \mathbf{x} \in B_j. \tag{4.7}$$

Using $|u(\mathbf{x})|^p = |u_g(\mathbf{x}') - \{u(\mathbf{x}) - u_g(\mathbf{x}')\}|^p \leq 2^p\{|u_g(\mathbf{x}') - u(\mathbf{x})|^p + |u_g(\mathbf{x}')|^p\}$, (4.7), and (4.6), we obtain the following:

$$\begin{aligned}
&\int_{B_j}\int_{B_{j-1}} \chi_\delta(\mathbf{x},\mathbf{x}')\,|u_g(\mathbf{x}') - u(\mathbf{x})|^p\ d\mathbf{x}'d\mathbf{x} \\
&\geq \frac{1}{2^p}\int_{B_j}\int_{B_{j-1}} \chi_\delta(\mathbf{x},\mathbf{x}')\,|u(\mathbf{x})|^p\ d\mathbf{x}'d\mathbf{x} - \int_{B_j}\int_{B_{j-1}} \chi_\delta(\mathbf{x},\mathbf{x}')\,|u_g(\mathbf{x}')|^p\ d\mathbf{x}'d\mathbf{x} \\
&= \frac{1}{2^p}\int_{B_j}\left\{\int_{B_{j-1}} \chi_\delta(\mathbf{x},\mathbf{x}')\ d\mathbf{x}'\right\}|u(\mathbf{x})|^p\ d\mathbf{x} - \int_{B_{j-1}}\left\{\int_{B_j} \chi_\delta(\mathbf{x},\mathbf{x}')d\mathbf{x}\right\}|u_g(\mathbf{x}')|^p\ d\mathbf{x}' \\
&= \frac{1}{2^p}|B_{j-1}|\int_{B_j}|u(\mathbf{x})|^p\ d\mathbf{x} - |B_j|\int_{B_{j-1}}|u_g(\mathbf{x}')|^p\ d\mathbf{x}' \\
&\geq \frac{c_{in}}{2^p}\,\delta^m\int_{B_j}|u(\mathbf{x})|^p\ d\mathbf{x} -\ c_{out}\ \delta^m\int_{B_{j-1}}|u_g(\mathbf{x}')|^p\ d\mathbf{x}'.
\end{aligned}$$

Hence,

$$\frac{c_{in}}{2^p}\ \delta^m\int_{B_j}|u(\mathbf{x})|^p\ d\mathbf{x} \leq a^{(p)}(u_g,u) + c_{out}\ \delta^m\int_{B_{j-1}}|u_g(\mathbf{x}')|^p\ d\mathbf{x}'.$$

For the cases $j = 1, 2$, we respectively have:

$$\frac{c_{in}}{2^p}\ \delta^m\int_{B_1}|u(\mathbf{x})|^p\ d\mathbf{x} \leq a^{(p)}(u_g,u) + c_{out}\ \delta^m\int_{B_0}|g(\mathbf{x}')|^p\ d\mathbf{x}' \tag{4.8}$$

$$\frac{c_{in}}{2^p}\ \delta^m\int_{B_2}|u(\mathbf{x})|^p\ d\mathbf{x} \leq a^{(p)}(u_g,u) + c_{out}\ \delta^m\int_{B_1}|u(\mathbf{x}')|^p\ d\mathbf{x}'. \tag{4.9}$$

In order to relate (4.9) to RHS of (4.8), multiply (4.8) by $c_{out}\ \frac{2^p}{c_{in}}$:

$$c_{out}\ \delta^m\int_{B_1}|u(\mathbf{x})|^p\ d\mathbf{x} \leq \frac{c_{out}\ 2^p}{c_{in}}\ a^{(p)}(u_g,u) + \frac{c_{out}^2\ 2^p}{c_{in}}\ \delta^m\int_{B_0}|g(\mathbf{x}')|^p\ d\mathbf{x}'. \tag{4.10}$$

Then, using (4.10), (4.9) becomes

$$\begin{aligned}
\frac{c_{in}}{2^p}\ \delta^m\int_{B_2}|u(\mathbf{x})|^p\ d\mathbf{x} &\leq a^{(p)}(u_g,u) + c_{out}\ \delta^m\int_{B_1}|u(\mathbf{x}')|^p\ d\mathbf{x}' \\
&\leq \{\frac{c_{out}\ 2^p}{c_{in}} + 1\}\ a^{(p)}(u_g,u) + \{\frac{c_{out}^2\ 2^p}{c_{in}} + c_{out}\}\ \delta^m\int_{B_0}|g(\mathbf{x}')|^p\ d\mathbf{x}'.
\end{aligned}$$

Repeating this procedure for $j = 3, \ldots, l$, using the fact that the covering of Ω is composed of disjoint strips, i.e., $\Omega = \cup_{k=1}^{l} B_k, \quad B_j \cap B_k = \emptyset,\ j \neq k$, we arrive at the result. $\square$

REMARK 4.2. *For the condition number analysis, the underlying function space is chosen to be $L_2(\Omega)$. In addition, we enforce a zero Dirichlet boundary condition. Hence, the nonlocal Poincaré inequality (4.2) reduces to the following:*

$$\lambda_{Pncr,\,2}\,\delta^m\,\|u\|^2_{L_2(\Omega)} \leq a(u,u), \tag{4.11}$$

where $a(u,u) := a^{(2)}(u,u)$, and $\lambda_{Pncr,\,2} = \lambda_{Pncr,\,2}(m,\Omega) > 0$ with $m \geq 1$.

REMARK 4.3. *When a general kernel function C is used, the Poincaré constant depends on that function as well, i.e., $\lambda_{Pncr} = \lambda_{Pncr}(m,p,\Omega,C)$. This is the result given in [4, Prop. 2.5]. With the canonical kernel function in (4.1), the Poincaré constant reduces to $\lambda_{Pncr} = \lambda_{Pncr}(m,p,\Omega)$. Our main contribution is the incorporation of the explicit δ-characterization. This is the key step used in the analysis of the condition number and its dependence on the spatial dimension; see §4.2 and §4.5.*

4.2. The choice of Poincaré parameter m and convergence to a local problem. After establishing a nonlocal Poincaré inequality with δ-dependence explicitly quantified, the big question becomes, *What is m in* (4.11)? The δ-quantification helps to answer that question, because we numerically observe that the value of m varies with the spatial dimension d; see Section 4.5. Next, we discuss a strategy to determine the value of m. This strategy is based on the fact that as $\delta \to 0$, a nonlocal bilinear form should converge to its corresponding local (classical) bilinear form. For discussions of convergence of other nonlocal operators to their classical local counterparts, see [16, 36]. We resort to a Taylor series expansion after enforcing a sufficient regularity assumption. In fact, with $u \in C^4(\overline{\Omega})$, we recover the numerically observed value of $m = 3$ for $d = 1$. We split the bilinear form in (3.5) into three pieces and introduce the change of variable in (3.6):

$$\begin{aligned} a(u,u) &= \frac{1}{2}\int_\Omega \left\{ \int_{-\delta}^{\delta} u^2(x+\varepsilon)\ d\varepsilon \right\}\ dx + \delta \int_\Omega u^2(x)\ dx - \int_\Omega \left\{ \int_{-\delta}^{\delta} u(x+\varepsilon)\ d\varepsilon \right\} u(x)\ dx \\ &=: T^{(1)} + T^{(2)} + T^{(3)}. \end{aligned}$$

For the integrands, we use the following Taylor expansions:

$$u(x+\varepsilon) = u(x) + \frac{\varepsilon}{1!}\frac{du}{dx} + \frac{\varepsilon^2}{2!}\frac{d^2u}{dx^2} + \frac{\varepsilon^3}{3!}\frac{d^3u}{dx^3} + \mathcal{O}(\varepsilon^4) \tag{4.12}$$

$$\begin{aligned} u^2(x+\varepsilon) = u^2(x) &+ \frac{2\,\varepsilon}{1!}\,u\,\frac{du}{dx} + \frac{2\,\varepsilon^2}{2!}\left\{ \left(\frac{du}{dx}\right)^2 + u\,\frac{d^2u}{dx^2} \right\} \\ &+ \frac{2\,\varepsilon^3}{3!}\left\{ 3\,\frac{du}{dx}\frac{d^2u}{dx^2} + u\,\frac{d^3u}{dx^3} \right\} + \mathcal{O}(\varepsilon^4). \end{aligned} \tag{4.13}$$

Using (4.12) and (4.13), we obtain the following expressions for the integrals:

$$T^{(1)} = \int_\Omega \left[\delta\,u^2 + \frac{2\,\delta^3}{3!}\left\{ \left(\frac{du}{dx}\right)^2 + u\,\frac{d^2u}{dx^2} \right\} + \mathcal{O}(\delta^5) \right]\ dx \tag{4.14}$$

$$T^{(2)} = \delta \int_\Omega u^2\ dx \tag{4.15}$$

$$T^{(3)} = -\int_\Omega \left[2\,\delta\,u^2 + \frac{2\,\delta^3}{3!}\,u\,\frac{d^2u}{dx^2} + \mathcal{O}(\delta^5) \right]\ dx \tag{4.16}$$

Combining (4.14), (4.15), and (4.16), denoting the local bilinear form by

$$\ell(u,u) := |u|^2_{H^1(\Omega)},$$

we obtain the following:

$$\begin{aligned} a(u,u) &= \int_\Omega \left[\frac{\delta^3}{3} \left(\frac{du}{dx}\right)^2 + \mathcal{O}(\delta^5) \right] dx \\ &= \frac{\delta^3}{3}\ \ell(u,u) + \mathcal{O}(\delta^5). \end{aligned}$$

Therefore, the scaled nonlocal bilinear form asymptotically converges to the local bilinear form:

$$3\ \delta^{-3}\ a(u,u) = \ell(u,u) + \mathcal{O}(\delta^2). \tag{4.17}$$

Using the nonlocal Poincaré inequality (4.11) and (4.17), we have

$$\lim_{\delta\to 0}\ 3\ \lambda_{Pncr,2}\ \delta^{m-3}\ \|u\|^2_{L_2(\Omega)} \le \ell(u,u).$$

Therefore, for the left hand side to remain finite, we have to enforce $m \ge 3$. We desire for the largest possible lower bound in the nonlocal Poincaré inequality, and $m = 3$ is the numerically observed value of m in 1D; see the experiments in §4.5.1.

4.3. An upper bound for $a(u,u)$. Without the need of a discretization of $a(u,u)$ as in the local case upper bound (i.e., inverse inequality), we prove the following dimension dependent estimate:

LEMMA 4.4. *Let $\Omega \subset \mathbb{R}^d$ be bounded. Then, there exists $\overline{\lambda} = \overline{\lambda}(\Omega) > 0$ such that*

$$a(u,u) \le \overline{\lambda}\ \delta^d\ \|u\|^2_{L_2(\Omega)}. \tag{4.18}$$

Proof. We provide the proof in 3D. The lower dimensional cases follow easily. First, we utilize a uniform partition of Ω composed of disjoint cubes of edge size δ. Namely,

$$\Omega \subseteq \bigcup_{(i,j,k)=(1,1,1)}^{(l_i,l_j,l_k)} b_{(i,j,k)}.$$

Hence, $|b_{(i,j,k)}| = \delta^3$. For a fixed δ, l_i, l_j, and l_k are functions of Ω only. The proof relies on the important observation that

$$\{\mathbf{x}' :\ \ \|\mathbf{x}-\mathbf{x}'\| \le \delta,\ \ \mathbf{x} \in b_{(i,j,k)}\} \subset \sum_{(ii,jj,kk)=(i-1,j-1,k-1)}^{(i+1.j+1,k+1)} b_{(ii,jj,kk)}. \tag{4.19}$$

For ease of notation, we replace the canonical kernel $\chi_\delta(\mathbf{x},\mathbf{x}')$ by 1 in its support due to (4.1). Furthermore, we introduce the characteristic function χ_Ω to avoid integration outside of Ω for the pieces of the cubes that may stay outside of Ω.

We split the bilinear form in (3.5) into three pieces:

$$
\begin{aligned}
a(u,u) &= \frac{1}{2}\int_{\Omega}\left\{\int_{\|\mathbf{x}-\mathbf{x}'\|\le\delta} |u(\mathbf{x}')-u(\mathbf{x})|^2 \, d\mathbf{x}'\right\} \, d\mathbf{x} \\
&\le \frac{1}{2}\int_{\Omega}\left\{\int_{\|\mathbf{x}-\mathbf{x}'\|\le\delta} |u(\mathbf{x}')|^2 \, d\mathbf{x}'\right\} \, d\mathbf{x} + \frac{1}{2}\int_{\Omega} |u(\mathbf{x})|^2 \left\{\int_{\|\mathbf{x}-\mathbf{x}'\|\le\delta} d\mathbf{x}'\right\} \, d\mathbf{x} \\
&\quad + \int_{\Omega}\left\{\int_{\|\mathbf{x}-\mathbf{x}'\|\le\delta} |u(\mathbf{x}')| \, d\mathbf{x}'\right\} |u(\mathbf{x})| \, d\mathbf{x} \\
&=: I^{(1)} + I^{(2)} + I^{(3)}.
\end{aligned}
$$

First, let

$$
\begin{aligned}
I^{(1)}_{(i,j,k)} &:= \frac{1}{2}\int_{b_{(i,j,k)}}\left\{\int_{\|\mathbf{x}-\mathbf{x}'\|\le\delta} |u(\mathbf{x}')|^2 \, d\mathbf{x}'\right\} \, d\mathbf{x} \\
I^{(2)}_{(i,j,k)} &:= \frac{1}{2}\int_{b_{(i,j,k)}} |u(\mathbf{x})|^2 \left\{\int_{\|\mathbf{x}-\mathbf{x}'\|\le\delta} d\mathbf{x}'\right\} \, d\mathbf{x} \\
I^{(3)}_{(i,j,k)} &:= \int_{b_{(i,j,k)}}\left\{\int_{\|\mathbf{x}-\mathbf{x}'\|\le\delta} |u(\mathbf{x}')| \, d\mathbf{x}'\right\} |u(\mathbf{x})| \, d\mathbf{x}.
\end{aligned}
$$

Then,

$$
I^{(1)} + I^{(2)} + I^{(3)} = \sum_{(i,j,k)=(1,1,1)}^{(l_i,l_j,l_k)} I^{(1)}_{(i,j,k)} + I^{(2)}_{(i,j,k)} + I^{(3)}_{(i,j,k)}.
$$

We will give an upper bound for each piece. Using (4.19), we immediately see:

$$
\begin{aligned}
I^{(1)}_{(i,j,k)} &\le \frac{1}{2}\int_{b_{(i,j,k)}}\left\{\sum_{(ii,jj,kk)=(i-1,j-1,k-1)}^{(i+1.j+1,k+1)} \int_{b_{(ii,jj,kk)}} |\chi_{\Omega}(\mathbf{x}')\, u(\mathbf{x}')|^2 \, d\mathbf{x}'\right\} \, d\mathbf{x} \\
&\le \frac{|b_{(i,j,k)}|}{2}\left\{\sum_{(ii,jj,kk)=(i-1,j-1,k-1)}^{(i+1.j+1,k+1)} \|\chi_{\Omega}\, u\|^2_{L_2(b_{(ii,jj,kk)})}\right\}.
\end{aligned}
$$

Hence, we obtain an upper bound for $I^{(1)}$by the following sum.

$$
I^{(1)} = \sum_{(i,j,k)=(1,1,1)}^{(l_i,l_j,l_k)} I^{(1)}_{(i,j,k)} \le c^{(1)}\ \delta^3\ \|u\|^2_{L_2(\Omega)}.
$$

Using (4.19), we immediately have:

$$
\begin{aligned}
I^{(2)}_{(i,j,k)} &\le \frac{1}{2} \int_{b_{(i,j,k)}} |\chi_\Omega(\mathbf{x})\, u(\mathbf{x})|^2 \; d\mathbf{x} \left\{ \sum_{(ii,jj,kk)=(i-1,j-1,k-1)}^{(i+1.j+1,k+1)} \int_{b_{(ii,jj,kk)}} \; d\mathbf{x}' \right\} \\
&= \frac{1}{2} \int_{b_{(i,j,k)}} |\chi_\Omega(\mathbf{x})\, u(\mathbf{x})|^2 \; d\mathbf{x} \left\{ \sum_{(ii,jj,kk)=(i-1,j-1,k-1)}^{(i+1.j+1,k+1)} |b_{(ii,jj,kk)}| \right\} \\
&= \frac{1}{2} \, \|\chi_\Omega \, u\|^2_{b_{(i,j,k)}} \; \left\{ \sum_{(ii,jj,kk)=(i-1,j-1,k-1)}^{(i+1.j+1,k+1)} \delta^3 \right\}
\end{aligned}
$$

Hence, we obtain the upper bound for $I^{(2)}$ by the following sum.

$$
I^{(2)} = \sum_{(i,j,k)=(1,1,1)}^{(l_i,l_j,l_k)} I^{(2)}_{(i,j,k)} \le c^{(2)} \; \delta^3 \; \|u\|^2_{L_2(\Omega)}.
$$

Finally, using (4.19), Hölder's inequality for each term and the identity $X\,Y \le \frac{1}{2}\{X^2 + Y^2\}$, we obtain the following:

$$
\begin{aligned}
I^{(3)}_{(i,j,k)} &\le \int_{b_{(i,j,k)}} |\chi_\Omega(\mathbf{x})\, u(\mathbf{x})| \; d\mathbf{x} \left\{ \sum_{(ii,jj,kk)=(i-1,j-1,k-1)}^{(i+1.j+1,k+1)} \int_{b_{(ii,jj,kk)}} |\chi_\Omega(\mathbf{x}')\, u(\mathbf{x}')| \; d\mathbf{x}' \right\} \\
&\le |b_{(i,j,k)}|^{1/2} \|\chi_\Omega \, u\|_{L_2(b_{(i,j,k)})} \; \left\{ \sum_{(ii,jj,kk)=(i-1,j-1,k-1)}^{(i+1.j+1,k+1)} |b_{(ii,jj,kk)}|^{1/2} \|\chi_\Omega \, u\|_{L_2(b_{(ii,jj,kk)})} \right\} \\
&= \delta^3 \; \left\{ \sum_{(ii,jj,kk)=(i-1,j-1,k-1)}^{(i+1.j+1,k+1)} \|\chi_\Omega \, u\|_{L_2(b_{(i,j,k)})} \; \|\chi_\Omega \, u\|_{L_2(b_{(ii,jj,kk)})} \right\} \\
&\le \frac{\delta^3}{2} \; \left\{ \sum_{(ii,jj,kk)=(i-1,j-1,k-1)}^{(i+1.j+1,k+1)} \|\chi_\Omega \, u\|^2_{L_2(b_{(i,j,k)})} + \|\chi_\Omega \, u\|^2_{L_2(b_{(ii,jj,kk)})} \right\}
\end{aligned}
$$

Hence, we obtain an upper bound for $I^{(3)}$ by the following sum.

$$
I^{(3)} = \sum_{(i,j,k)=(1,1,1)}^{(l_i,l_j,l_k)} I^{(3)}_{(i,j,k)} \le c^{(3)} \; \delta^3 \; \|u\|^2_{L_2(\Omega)}.
$$

The result follows from adding the upper bounds for the three pieces. ∎

Remark 4.5. *The upper bound is* numerically sharp *when a piecewise constant discretization of $a(\cdot,\cdot)$ is used; see* §*4.5*.

4.4. The Conditioning of the Stiffness Matrix $\boldsymbol{K}$. Combining the nonlocal Poincaré inequality (4.11) and the upper bound (4.18), we arrive at the condition number estimate:

Theorem 4.1. *The following spectral equivalence holds:*

$$
\lambda_{Pncr,\,2} \; \delta^m \le \frac{a(u,u)}{\|u\|^2_{L_2(\Omega)}} \le \overline{\lambda} \; \delta^d. \tag{4.20}
$$

Let K be the stiffness matrix produced by the discretized $a(u, u)$. Then, the condition number of K has the following bound:

$$\kappa(K) \lesssim \delta^{d-m}. \tag{4.21}$$

(a) Constant δ, vary h.

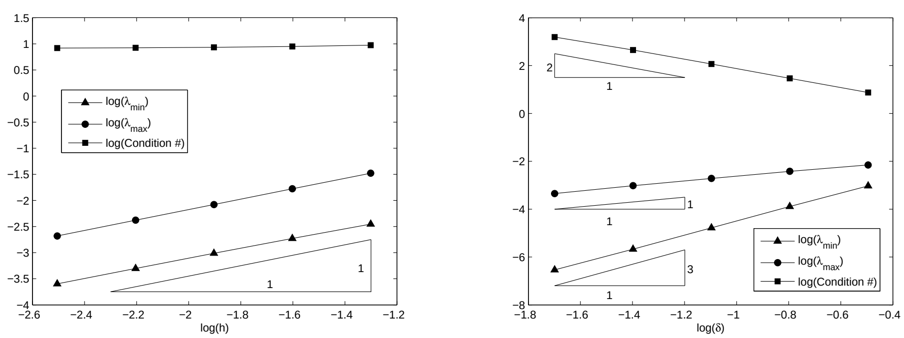
$1/h$	δ	$\lambda_{\min} \neq 0$	$\lambda_{\max}$	Condition #
20	0.3	3.53E-03	3.32E-02	9.41E+00
40	0.3	1.87E-03	1.67E-02	8.92E+00
80	0.3	9.74E-04	8.36E-03	8.58E+00
160	0.3	4.99E-04	4.18E-03	8.39E+00
320	0.3	2.52E-04	2.09E-03	8.29E+00

(b) Constant h, vary δ.

$1/h$	δ	$\lambda_{\min} \neq 0$	$\lambda_{\max}$	Condition #
100	0.02	2.90E-07	4.50E-04	1.55E+03
100	0.04	2.13E-06	9.54E-04	4.49E+02
100	0.08	1.66E-05	1.93E-03	1.16E+02
100	0.16	1.29E-04	3.80E-03	2.93E+01
100	0.32	9.36E-04	7.03E-03	7.51E+00

TABLE 4.1

Condition number for K in 1D for (a) constant δ, allowing h to vary, and (b) constant h, allowing δ to vary. This data is plotted in Figure 4.1.



(a) Constant δ, vary h. (b) Constant h, vary δ.

FIG. 4.1. *Condition number for K in 1D for (a) constant δ, allowing h to vary, and (b) constant h, allowing δ to vary. The condition number is h-independent, but varies with δ^{-2}. These figures are plotted from data in Table 4.1.*

4.5. Numerical Verification of Condition Number by a Finite Element Formulation. For all computational results in this paper, we assume that $\Omega = [0, 1]^d$ is the unit d-cube with a conforming triangulation $\mathcal{T}_h$ where each element K of $\mathcal{T}_h$ is a d-cube with a side length $h > 0$, where d is the spatial dimension. Consequently, each element in 1D, 2D, and 3D is a line segment of length h, a square of area h^2, and a cube of volume h^3, respectively. Let $V_h \subset V$ be a finite dimensional subspace of V from (3.1). We use a Galerkin finite element formulation of (3.2):

$$a(u_h, v_h) = (b_h, v_h) \qquad \forall v_h \in V_h.$$

(a) Constant δ, vary h.

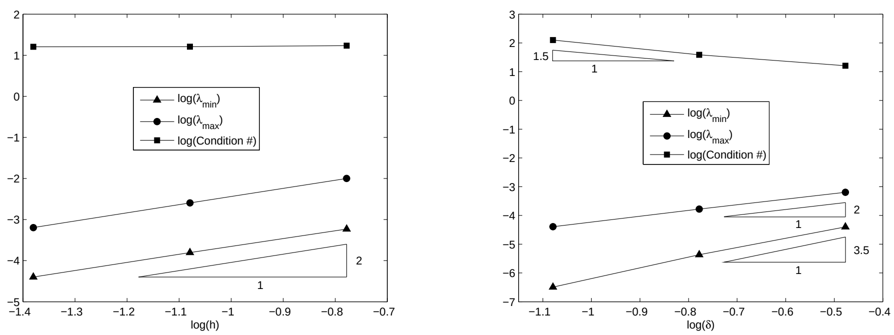
$1/h$	δ	$\lambda_{\min} \neq 0$	$\lambda_{\max}$	Condition #
6	0.333	5.86E-04	1.00E-02	1.71E+01
12	0.333	1.57E-04	2.54E-03	1.62E+01
24	0.333	3.98E-05	6.40E-04	1.61E+01

(b) Constant h, vary δ.

$1/h$	δ	$\lambda_{\min} \neq 0$	$\lambda_{\max}$	Condition #
24	0.083	3.22E-07	4.05E-05	1.26E+02
24	0.166	4.32E-06	1.67E-04	3.86E+01
24	0.333	3.98E-05	6.40E-04	1.61E+01

TABLE 4.2

Condition number for K in 2D for (a) constant δ, allowing h to vary, and (b) constant h, allowing δ to vary. This data is plotted in Figure 4.1.



(a) Constant δ, vary h. (b) Constant h, vary δ.

FIG. 4.2. *Condition number for K in 2D for (a) constant δ, allowing h to vary, and (b) constant h, allowing δ to vary. The condition number is h-independent, but varies with $\delta^{-3/2}$. These figures are plotted from data in Table 4.1.*

For convenience, we have used Dirichlet boundary value for the theoretical construction. To verify our theoretical results numerically, for implementation convenience, we utilize the discretization of a pure Neumann boundary value problem.

$$a(u_h, v_h) = 0 \qquad \forall v_h \in V_h, \tag{4.22}$$

where we take V_h to be the space of piecewise constant shape functions on the mesh $\mathcal{T}_h$. We denote the resulting stiffness matrix by K. We numerically determine the largest and smallest nonzero eigenvalues, defining the effective condition number of the problem.

4.5.1. Results in One Dimension. Results in this section appear in Tables 4.1 and Figures 4.1. We first compute the condition number of K for different h while holding δ constant to verify our conjecture that the condition number of K is h-independent. The minimum nonzero and the maximum eigenvalues depend linearly on h, with a slope of unity. Consequently, the condition number of K is mesh size independent. We then compute the condition number of K for different δ while holding h constant, and observe that the condition number varies with δ^{-2}. Further, the maximum eigenvalue is proportional to δ^d ($d = 1$ in this case), in agreement with Lemma 4.4. Lastly, the minimum eigenvalue varies as δ^3, in agreement with our

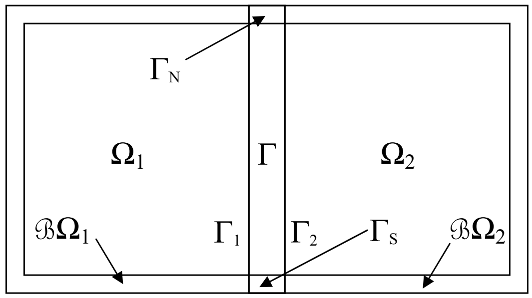


FIG. 5.1. *Nonlocal two-subdomain problem. Note that the interface* Γ *is* $d = 2$*-dimensional.*

finding of $m = 3$ in §4.2. This suggests that, in one dimension, we should redefine $C(x, x')$ in (4.1) as

$$C(x, x') = \left\{ \begin{array}{ll} \delta^{-3}, & \|x - x'\| \le \delta \\ 0, & \text{otherwise.} \end{array} \right. ,$$

for consistency with the weak form of the classical (local) Laplace operator in the limit $\delta \to 0$.

It is instructive to compare convergence results for the nonlocal problem with the classical convergence results for the (local) discrete Laplace equation. For the discrete local Laplace equation, the condition number varies with h^{-2} [40, Theorem B.32]. In light of these numerical experiments, it appears that the characteristic length scale δ of the nonlocal problem (1.1) plays the role that the mesh size h plays in the discrete (local) Laplace equation. This is interesting, as δ is a parameter of the continuum model, whereas h is a parameter of the discrete model.

4.5.2. Results in Two Dimensions. Results in this section appear in Tables 4.2 and Figures 4.2. We first compute the condition number of K for different h while holding δ constant to verify our conjecture that the condition number of K is h-independent. The minimum and maximum eigenvalues depend linearly on h, with a slope of two, and again the condition number of K is mesh size independent. We then compute the condition number of K for different δ while holding h constant, and observe that the condition number varies with $\delta^{-3/2}$. Further, the maximum eigenvalue is proportional to δ^d ($d = 2$ in this case), in agreement with Lemma 4.4.

5. A Nonlocal Two-Domain Problem. We will construct both strong and weak (variational) formulations for domain decomposition. We first identify the pieces of the domain for this decomposition. Consider the domain in Figure 5.1. The nonlocal boundary of Ω, $\mathcal{B}\Omega$, is defined to be the closed region of thickness δ surrounding Ω. Let Γ be the open region corresponding to the interface between the two overlapping open subdomains $\Omega^{(1)}$ and $\Omega^{(2)}$. We define the overlapping subdomains $\Omega^{(i)}, \;\; i = 1, 2$, as the following:

$$\Omega^{(i)} := \Omega_i \cup \Gamma \cup \Gamma_i,$$

where Γ_i is the open line segment adjacent to Ω_i and Γ. Let $\mathcal{B}\Omega_i$ be the nonlocal closed boundary of Ω_i that intersects $\mathcal{B}\Omega$. In addition, the remaining pieces Γ_N and Γ_S are the open regions to the north and the south of Γ, respectively. The nonlocal boundary of $\Omega^{(i)}$ is defined as follows:

$$\mathcal{B}\Omega^{(i)} := \mathcal{B}\Omega_i \cup \overline{\Gamma_N} \cup \overline{\Gamma_S}.$$

The three equivalence results we will prove next are the main domain decomposition contributions of this article:

1. Equivalence of the one-domain strong and two-domain strong forms.
2. Equivalence of the one-domain weak and two-domain weak forms.
3. Equivalence of the two-domain strong and two-domain weak forms.

The fact that the main problem (1.1) is an integral equation provides a convenient framework for establishing these equivalences, especially for the last item.

5.1. Two-Domain Strong Form. We seek a solution to (1.1) in $L_2(\Omega)$, i.e., $u \in L_2(\Omega)$. We present a two-domain strong formulation of (1.1) and its equivalence to the original single-domain formulation (1.1). Let $u^{(i)}(\mathbf{x}) := \chi_{\Omega^{(i)}}(\mathbf{x})\, u(\mathbf{x})$. For the sake of brevity, we utilize the following notation:

$$\mathcal{L}_{\Omega_i}(u)(\mathbf{x}) := -\chi_{\Omega_i}(\mathbf{x}) \int_{\Omega^{(i)}} \chi_\delta(\mathbf{x}, \mathbf{x}') \cdot [u(\mathbf{x}') - u(\mathbf{x})]\; d\mathbf{x}' \tag{5.1}$$

$$\mathcal{L}_{\overline{\Gamma}}(u)(\mathbf{x}) := -\; \chi_{\overline{\Gamma}}(\mathbf{x}) \int_{\Omega} \chi_\delta(\mathbf{x}, \mathbf{x}') \cdot [u(\mathbf{x}') - u(\mathbf{x})]\; d\mathbf{x}'. \tag{5.2}$$

The two-domain version of (1.1) may be stated as follows: For $i = 1, 2$,

$$\mathcal{L}_{\Omega_i}(u^{(i)})(\mathbf{x}) = \chi_{\Omega_i}(\mathbf{x})\; b(\mathbf{x}), \tag{5.3a}$$

$$\chi_{\overline{\Gamma}}(\mathbf{x})\; u^{(1)} = \chi_{\overline{\Gamma}}(\mathbf{x})\; u^{(2)}, \tag{5.3b}$$

$$\sum_{i=1,2} \frac{1}{2}\; (\mathbf{x})\; \mathcal{L}_{\overline{\Gamma}}(u^{(i)})(\mathbf{x}) = \chi_{\overline{\Gamma}}(\mathbf{x})\; b(\mathbf{x}). \tag{5.3c}$$

Remark 5.1. *In light of* (5.3)*, we make the following crucial observation about nonlocal domain decomposition. The strong version of the two-domain decomposition turns into a three-domain decomposition in which we produce three domain conditions, i.e.,* (5.3a) *for* $i = 1, 2$*, and* (5.3c)*, together with the transmission condition* (5.3b)*.*

We now show that the two-domain strong form is equivalent to the one-domain strong form.

Lemma 5.2. *The problems* (1.1) *and* (5.3) *are equivalent.*

Proof. (1.1) $\Rightarrow$ (5.3) :
Let $u^{(i)} = u|_{\Omega^{(i)}} \in V^{(i)}$, where $V^{(i)}$ is defined in (5.7). Using the nonoverlapping decomposition of the domain, we have:

$$\chi_\Omega(\mathbf{x}) = \chi_{\Omega_1}(\mathbf{x}) + \chi_{\Omega_2}(\mathbf{x}) + \chi_{\overline{\Gamma}}(\mathbf{x}). \tag{5.4}$$

Using (5.4), suppressing the $\mathbf{x}$-dependence,

$$\begin{aligned}\mathcal{L}(u) &= \chi_{\Omega_1}\; \mathcal{L}(u) + \chi_{\Omega_2}\; \mathcal{L}(u) + \frac{1}{2}\; \chi_{\overline{\Gamma}}\; \mathcal{L}(u) + \frac{1}{2}\; \chi_{\overline{\Gamma}}\; \mathcal{L}(u) \\ &= \chi_{\Omega_1}\; \mathcal{L}_{\Omega_1}(u^{(1)}) + \chi_{\Omega_2}\; \mathcal{L}_{\Omega_2}(u^{(2)}) + \frac{1}{2}\; \mathcal{L}_{\overline{\Gamma}}(u^{(1)}) + \frac{1}{2}\; \mathcal{L}_{\overline{\Gamma}}(u^{(2)}) = b. \end{aligned} \tag{5.5}$$

Multiplying (5.5) by χ_{Ω_i}, we get (5.3a):

$$\chi_{\Omega_i}\; \mathcal{L}_{\Omega_i}(u^{(i)}) = \chi_{\Omega_i}\; b.$$

Likewise, multiplying (5.5) by $\chi_{\overline{\Gamma}}$, we get (5.3c):

$$\frac{1}{2}\; \mathcal{L}_{\overline{\Gamma}}(u^{(1)}) + \frac{1}{2}\; \mathcal{L}_{\overline{\Gamma}}(u^{(2)}) = \chi_{\overline{\Gamma}}\; b.$$

Finally, we trivially obtain (5.3b) by the following:

$$u^{(1)}|_{\overline{\Gamma}} = u|_{\overline{\Gamma}} = u^{(2)}|_{\overline{\Gamma}}.$$

(5.3) ⇒ (1.1) :
Letting

$$u := \begin{cases} u^{(1)}, & \text{in } \Omega_1 \\ u^{(2)}, & \text{in } \Omega_2 \\ u^{(1)} = u^{(2)}, & \text{in } \overline{\Gamma}, \end{cases} \tag{5.6}$$

and adding (5.3a) and (5.3c), we obtain:

$$\begin{aligned} b &= \sum_{i=1,2} \left\{ \mathcal{L}_{\Omega_i}(u^{(i)}) + \frac{1}{2}\, \mathcal{L}_{\overline{\Gamma}}(u^{(i)}) \right\} \\ &= \sum_{i=1,2} \left\{ \mathcal{L}_{\Omega_i}(u) + \frac{1}{2}\, \mathcal{L}_{\overline{\Gamma}}(u) \right\} \\ &= \sum_{i=1,2} \left\{ \chi_{\Omega_i}\, \mathcal{L}(u) + \frac{1}{2}\, \chi_{\overline{\Gamma}}\, \mathcal{L}_{\overline{\Gamma}}(u) \right\} \\ &= \chi_{\Omega}\, \mathcal{L}(u). \end{aligned}$$

□

5.2. Two-Domain Variational Form. We present a two-domain weak formulation of (3.2) and its equivalence to the original single-domain formulation (3.2). We define the spaces, $i = 1, 2$,

$$\begin{aligned} V^{(i)} &:= \left\{ v^{(i)} \in L_2(\Omega^{(i)}) \ : \ v|_{\mathcal{B}\Omega^{(i)}} = 0 \right\}, \\ V^{(i),0} &:= \left\{ v \in L_2(\Omega^{(i)}) \ : \ v|_{\mathcal{B}\Omega^{(i)} \cup \Gamma \cup \Gamma_i} = 0 \right\}, \\ \Lambda &:= \left\{ \mu \in L_2(\Gamma) : \mu = v|_\Gamma \text{ for some suitable } v \in L_{2,0}(\Omega) \right\}, \end{aligned} \tag{5.7}$$

and the bilinear form $a_{\Omega^{(i)}}(u,v) : L_2(\Omega^{(i)}) \times L_2(\Omega^{(i)}) \to \mathbb{R}$ as follows:

$$\begin{aligned} a_{\Omega^{(i)}}(u,v) := &-\int_{\Omega_i} \left\{ \int_{\Omega^{(i)}} \chi_\delta(\mathbf{x},\mathbf{x}') \cdot [u(\mathbf{x}') - u(\mathbf{x})]\ d\mathbf{x}' \right\} \cdot v(\mathbf{x})\ d\mathbf{x} \\ &-\frac{1}{2} \int_{\Gamma} \left\{ \int_{\Omega} \chi_\delta(\mathbf{x},\mathbf{x}') \cdot [u(\mathbf{x}') - u(\mathbf{x})]\ d\mathbf{x}' \right\} \cdot v(\mathbf{x})\ d\mathbf{x}. \end{aligned} \tag{5.8}$$

Remark 5.3. *We have chosen the weights of $\frac{1}{2}$ in the definition of the bilinear form in (5.8). Such a choice typically indicates that the underlying body is homogeneous. For an inhomogeneous body, the weights can be chosen as a convex combination reflecting the inhomogeneity.*

We utilize the following notation to suppress the integrals in (5.8):

$$a_{\Omega_i}(u,v) := -\int_{\Omega_i} \left\{ \int_{\Omega^{(i)}} \chi_\delta(\mathbf{x},\mathbf{x}') \cdot [u(\mathbf{x}') - u(\mathbf{x})]\ d\mathbf{x}' \right\} \cdot v(\mathbf{x})\ d\mathbf{x} \tag{5.9}$$

$$a_{\Gamma}(u,v) := -\int_{\Gamma} \left\{ \int_{\Omega} \chi_\delta(\mathbf{x},\mathbf{x}') \cdot [u(\mathbf{x}') - u(\mathbf{x})]\ d\mathbf{x}' \right\} \cdot v(\mathbf{x})\ d\mathbf{x} \tag{5.10}$$

We can now represent the bilinear form (5.8) as:

$$a_{\Omega^{(i)}}(u,v) = \frac{1}{2}\, a_\Gamma(u,v) + a_{\Omega_i}(u,v).$$

REMARK 5.4. *The test function $v_i = v|_{\Omega_i} \in V^{(i),0},\ i=1,2$ has its support only in Ω_i not $\Omega^{(i)}$. Hence, we may reduce the bilinear form (5.8) to*

$$a_{\Omega^{(i)}}(u^{(i)}, v_i) = a_{\Omega_i}(u^{(i)}, v_i). \tag{5.11}$$

Although, $a_\Gamma(u^{(i)}, v_i)$ may appear to create a coupling between the subdomains, no such coupling exists because v_i vanishes on Γ. Therefore, subdomain condition (5.12a) is an expression only for subdomain $\Omega^{(i)}$.

Now, we state the two-domain weak form following the notation of [30]: Find $u^{(i)} \in V^{(i)}$, $i=1,2$:

$$a_{\Omega^{(i)}}(u^{(i)}, v_i) = (b, v_i)_{\Omega_i} \qquad \forall v_i \in V^{(i),0}, \tag{5.12a}$$

$$u^{(1)} = u^{(2)} \qquad \text{on } \overline{\Gamma}, \tag{5.12b}$$

$$\sum_{i=1,2} a_{\Omega^{(i)}}(u^{(i)}, \mathcal{R}^{(i)}\mu) = (b,\mu)_\Gamma + \sum_{i=1,2} (b, \mathcal{R}^{(i)}\mu)_{\Omega_i} \qquad \forall \mu \in \Lambda. \tag{5.12c}$$

where $\mathcal{R}^{(i)}$ denotes any possible extension operator from $\Gamma \cup \Gamma_i$ to $V^{(i)}$. Next, we will show that the one- and two-domain weak forms are equivalent. The proof for the local case can be found in [30, Lemma 1.2.1].

LEMMA 5.5. *The problems (3.2) and (5.12) are equivalent.*

Proof. (3.2) $\Rightarrow$ (5.12) :
Let $u^{(i)} = u|_{\Omega^{(i)}} \in V^{(i)}$ and $v_i = v|_{\Omega_i} \in V^{(i),0},\ i=1,2$. Extend these functions by zero extension;

$$\theta^{(i)}u^{(i)} := \begin{cases} u^{(i)}, & \text{in } \Omega^{(i)} \\ 0, & \text{otherwise} \end{cases}$$

$$\theta_i v_i := \begin{cases} v_i, & \text{in } \Omega_i \\ 0, & \text{otherwise.} \end{cases}$$

By LHS of (3.2) and using $v_i|_\Gamma = 0$:

$$\begin{aligned} a(\theta^{(i)}u^{(i)}, \theta_i v_i) &= -\int_\Omega \left\{ \int_\Omega \chi_\delta(\mathbf{x},\mathbf{x}') \cdot [\theta^{(i)}u^{(i)}(\mathbf{x}') - \theta^{(i)}u^{(i)}(\mathbf{x})]\ d\mathbf{x}' \right\} \cdot \theta_i v_i(\mathbf{x})\ d\mathbf{x} \\ &= a_{\Omega_i}(u^{(i)}, v_i) \\ &= \frac{1}{2}\, a_\Gamma(u^{(i)}, v_i) + a_{\Omega_i}(u^{(i)}, v_i) \\ &= a_{\Omega^{(i)}}(u^{(i)}, v_i) \end{aligned}$$

By RHS of (3.2),

$$(b, \theta_i v_i) = (b, v_i)_{\Omega_i}.$$

Hence, (5.12a) is satisfied. (5.12b) is trivially satisfied.

Further, for $\mu \in \Lambda$ define the function $\mathcal{R}\mu$ as:

$$\mathcal{R}\mu := \begin{cases} \mathcal{R}^{(1)}\mu, & \text{in } \Omega^{(1)} \\ \mathcal{R}^{(2)}\mu, & \text{in } \Omega^{(2)}. \end{cases}$$

Since $\mathcal{R}\mu$ lives only in $\Omega_1 \cup \Gamma_1 \cup \Gamma \cup \Gamma_2 \cup \Omega_2$, it vanishes on $\mathcal{B}\Omega$. Therefore, $\mathcal{R}\mu \in V$.

From (3.2), partitioning the outer integral and using $\mathcal{R}^{(1)}\mu = \mathcal{R}^{(2)}\mu = \mu$ on Γ, we obtain the LHS of (5.12c):

$$\begin{aligned} a(u, \mathcal{R}\mu) &= \frac{1}{2}\, a_\Gamma(u^{(1)}, \mu) + \frac{1}{2}\, a_\Gamma(u^{(2)}, \mu) + \sum_{i=1,2} a_{\Omega_i}(u^{(i)}, \mathcal{R}^{(i)}\mu) \\ &= \frac{1}{2}\, a_\Gamma(u^{(1)}, \mathcal{R}^{(1)}\mu) + \frac{1}{2}\, a_\Gamma(u^{(2)}, \mathcal{R}^{(2)}\mu) + \sum_{i=1,2} a_{\Omega_i}(u^{(i)}, \mathcal{R}^{(i)}\mu) \\ &= a^{(1)}(u^{(1)}, \mathcal{R}^{(1)}\mu) + a^{(2)}(u^{(2)}, \mathcal{R}^{(2)}\mu). \end{aligned}$$

Likewise, from (3.2) and partitioning the integral, we obtain the RHS of (5.12c):

$$(b, \mathcal{R}\mu)_\Omega = (b, \mathcal{R}^{(1)}\mu)_{\Omega_1} + (b, \mathcal{R}^{(2)}\mu)_{\Omega_2} + (b, \mu)_\Gamma.$$

Hence, we obtain the transmission condition (5.12c).

(5.12) $\Rightarrow$ (3.2) :
Let $u_\Gamma := u^{(1)}$ (due to (5.12b), we also have $u_\Gamma = u^{(2)}$) and

$$u := \begin{cases} u^{(1)}, & \text{in } \Omega_1 \\ u^{(2)}, & \text{in } \Omega_2 \\ u_\Gamma, & \text{in } \Gamma. \end{cases} \tag{5.13}$$

We partition the outer integral, use (5.13) and the transmission condition (5.12b). Then, for $v \in V$, LHS in (3.2) becomes the following:

$$\begin{aligned} a(u, v) &= \frac{1}{2}\, a_\Gamma(u, v) + \frac{1}{2}\, a_\Gamma(u, v) + \sum_{i=1,2} a_{\Omega_i}(u, v) \\ &= \frac{1}{2}\, a_\Gamma(u_\Gamma, v) + \frac{1}{2}\, a_\Gamma(u_\Gamma, v) + \sum_{i=1,2} a_{\Omega_i}(u^{(i)}, v) \\ &= \frac{1}{2}\, a_\Gamma(u^{(1)}, v) + \frac{1}{2}\, a_\Gamma(u^{(2)}, v) + \sum_{i=1,2} a_{\Omega_i}(u^{(i)}, v) \\ &= \sum_{i=1,2} a_{\Omega^{(i)}}(u^{(i)}, v). \end{aligned} \tag{5.14}$$

Let $\mu := v|_\Gamma$. Then, $v - \mathcal{R}^{(i)}\mu \in V^{(i),0}$. First, we add and subtract $\mathcal{R}^{(i)}\mu$ to (5.14) and apply the domain conditions (5.12a) for $v - \mathcal{R}^{(i)}\mu$. Then, we apply the transmission condition (5.12c) and use $v|_\Gamma = \mu$. Hence, we arrive at the RHS in (3.2):

$$\begin{aligned} \sum_{i=1,2} a_{\Omega^{(i)}}(u^{(i)}, v) &= \sum_{i=1,2} a_{\Omega^{(i)}}(u^{(i)}, v - \mathcal{R}^{(i)}\mu) + \sum_{i=1,2} a_{\Omega^{(i)}}(u^{(i)}, \mathcal{R}^{(i)}\mu) \\ &= \sum_{i=1,2} (b, v - \mathcal{R}^{(i)}\mu)_{\Omega_i} + \sum_{i=1,2} a_{\Omega^{(i)}}(u^{(i)}, \mathcal{R}^{(i)}\mu) \\ &= \sum_{i=1,2} (b, v - \mathcal{R}^{(i)}\mu)_{\Omega_i} + \sum_{i=1,2} (b, \mathcal{R}^{(i)}\mu)_{\Omega_i} + (b, \mu)_\Gamma \\ &= \sum_{i=1,2} (b, v)_{\Omega_i} + (b, \mu)_\Gamma \\ &= (b, v). \end{aligned}$$

□

5.3. Equivalence of Two-Domain Strong and Two-Domain Weak Forms. We prove the equivalence of the strong and weak forms of the two-domain formulation.

LEMMA 5.6. *The problems* (5.12) *and* (5.3) *are equivalent.*

Proof. (5.3) $\Rightarrow$ (5.12) :
Testing (5.3a) against $v_i \in V^{(i),0}$ and using $v_i|_{\overline{\Gamma}} = 0$, we obtain (5.12a):

$$\begin{aligned}(b, v_i)_{\Omega_i} &= (\chi_{\Omega_i}\ \mathcal{L}_{\Omega_i}(u^{(i)}), v_i)\\ &= a_{\Omega_i}(u^{(i)}, v_i)\\ &= a_{\Omega^{(i)}}(u^{(i)}, v_i).\end{aligned}$$

(5.3b) and (5.12b) are identical.

Next we show how to obtain (5.12c). Given $\mu \in \Lambda$. Testing (5.3a) against $\mathcal{R}^{(i)}\mu$, we immediately obtain:

$$a_{\Omega_i}(u^{(i)}, \mathcal{R}^{(i)}\mu) = (b, \mathcal{R}^{(i)}\mu)_{\Omega_i}. \tag{5.15}$$

Likewise, testing (5.3c) against μ, we get:

$$\frac{1}{2}\ (\chi_{\overline{\Gamma}}\ \mathcal{L}(u^{(1)}), \mu) + \frac{1}{2}\ (\chi_{\overline{\Gamma}}\ \mathcal{L}(u^{(2)}), \mu) = (\chi_{\overline{\Gamma}}\ b, \mu) = (b, \mu)_{\overline{\Gamma}}. \tag{5.16}$$

Since $\mathcal{R}^{(i)}$ is any extension operator, $\mathcal{R}^{(i)}\mu|_{\Gamma} = \mu$, we can rewrite (5.16) as follows:

$$\frac{1}{2}\ (\chi_{\overline{\Gamma}}\ \mathcal{L}(u^{(1)}), \mathcal{R}^{(1)}\mu) + \frac{1}{2}\ (\chi_{\overline{\Gamma}}\ \mathcal{L}(u^{(2)}), \mathcal{R}^{(2)}\mu) = (b, \mu)_{\overline{\Gamma}}. \tag{5.17}$$

Adding (5.15) and (5.17), we obtain (5.12c).

(5.12) $\Rightarrow$ (5.3) :
Since for $v_i \in V^{(i),0}$, we have $v_i|_{\overline{\Gamma}} = 0$, (5.12a) reduces to the following:

$$\begin{aligned}(b, v_i)_{\Omega_i} &= a_{\Omega_i}(u^{(i)}, v_i)\\ &= (\mathcal{L}_{\Omega_i}(u^{(i)}), v_i).\end{aligned}$$

Equivalently, we have

$$(\mathcal{L}_{\Omega_i}(u^{(i)}) - \chi_{\Omega_i}\, b,\ v_i) = 0 \qquad \forall v_i \in V^{(i),0}.$$

We obtain (5.3a).

Since we have (5.3a), test it against $\mathcal{R}^{(i)}\mu$:

$$\begin{aligned}(\mathcal{L}_{\Omega_i}(u^{(i)}), \mathcal{R}^{(i)}\mu) &= (\chi_{\Omega_i}\, b, \mathcal{R}^{(i)}\mu)\\ a_{\Omega_i}(u^{(i)}, \mathcal{R}^{(i)}\mu) &= (b, \mathcal{R}^{(i)}\mu)_{\Omega_i}.\end{aligned} \tag{5.18}$$

Subtracting (5.18) from (5.12c), we obtain

$$\frac{1}{2}\ a_{\Gamma}(u^{(1)}, \mathcal{R}^{(1)}\mu) + \frac{1}{2}\ a_{\Gamma}(u^{(2)}, \mathcal{R}^{(2)}\mu) = (b, \mu)_{\Gamma}.$$

Using $\mathcal{R}^{(i)}\mu|_{\Gamma} = \mu$ for the above equation, we arrive at:

$$(\frac{1}{2}\ \chi_{\overline{\Gamma}}\ \mathcal{L}(u^{(1)}) + \frac{1}{2}\ \chi_{\overline{\Gamma}}\ \mathcal{L}(u^{(2)}) - \chi_{\overline{\Gamma}}\, b,\ \mu) = 0 \qquad \forall \mu \in \Lambda.$$

We obtain (5.3c). $\square$

6. Towards Nonlocal Substructuring. Here we write out the linear algebraic representations arising from the two-subdomain weak form (5.12), identifying the discrete subdomain equations and transmission conditions. We then construct a nonlocal Schur complement, discuss its condition number as a function of h, δ, and provide supporting numerical experiments.

6.1. Linear Algebraic Representations. We consider a finite element discretization of (5.12). Letting $V_h^{(i)}$ denote the finite element space corresponding to $\Omega^{(i)}$, we define:

$$\begin{aligned} V_h^{(i),0} &:= \left\{ v_h \in V_h^{(i)} : \ v_h|_{\mathcal{B}\Omega_i \cup \Gamma \cup \Gamma_i} = 0 \right\} \\ \Lambda_h &:= \{ \mu_h : \ \mu_h = v_h|_\Gamma \text{ for some suitable } v_h \in V_h \} . \end{aligned}$$

We see that the finite element formulation of (5.12) can be written as:

$$\begin{aligned} a_{\Omega^{(i)}}(u_h^{(i)}, v_{i,h}) &= (b, v_{i,h})_{\Omega_i} & \forall v_{i,h} \in V_h^{(i),0}, \\ u_h^{(1)} &= u_h^{(2)} & \text{on } \overline{\Gamma}, \\ \sum_{i=1,2} a_{\Omega^{(i)}}(u_h^{(i)}, \mathcal{R}_h^{(i)} \mu_h) &= (b, \mu_h)_\Gamma + \sum_{i=1,2} (b, \mathcal{R}_h^{(i)} \mu_h)_{\Omega_i} & \forall \mu_h \in \Lambda_h. \end{aligned}$$

where $\mathcal{R}_h^{(i)}$ denotes any possible extension operator from $\Gamma_h \cup \Gamma_{i,h}$ to $V_h^{(i)}$. Following standard practice, we take these extension operators to be the finite element interpolant, which is defined to be equal to μ_h at the nodes in the thick interface Γ and zero on the internal nodes of Ω_i. If we number nodes in Ω_1 first, nodes in Ω_2 second, and nodes in Γ last, we will arrive at a global stiffness matrix that takes the traditional block arrowhead form:

$$K = \begin{bmatrix} K_{11} & 0 & K_{1\Gamma} \\ 0 & K_{22} & K_{2\Gamma} \\ K_{\Gamma 1} & K_{\Gamma 2} & K_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_\Gamma \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_\Gamma \end{bmatrix}. \tag{6.2}$$

6.2. Energy Minimizing Extension and the Schur Complement Conditioning. In order to study the conditioning of the Schur complement in the nonlocal setting, we first define an analog of the harmonic extension in the local case.

DEFINITION 6.1. *For a given $q \in L_2(\Gamma)$, $E_i(q) : L_2(\Gamma) \to L_{2,0}(\Omega^{(i)})$ is called its energy minimizing extension into Ω_i, if the following holds:*

$$E_i(q)|_\Gamma = q, \tag{6.3}$$

and $a(E_i(q), v) = 0$ for all $v \in L_{2,0}(\Omega^{(i)})$ such that $v|_\Gamma = 0$.

The energy minimizing extension $E_i(q)$ of q defines a canonical bilinear form $s_i(q, q) : L_2(\Gamma) \times L_2(\Gamma) \to \mathbb{R}$ that is associated to the interface Γ whose discretization corresponds to the subdomain Schur complement matrix $S^{(i)}$ below.

$$s_i(q, q) := a(E_i(q), E_i(q)) \tag{6.4}$$
$$\langle S^{(i)} q_h, q_h \rangle := a(E_i(q_h), E_i(q_h)). \tag{6.5}$$

Let us denote the restriction of $u \in L_{2,0}(\Omega^{(i)})$ to Γ by $u_\Gamma := u|_\Gamma$. The following discussion will reveal the reason why $E_i(u_\Gamma)$ is called an energy minimizing extension. Let us consider the following decomposition of u:

$$u = [u - E_i(u_\Gamma)] + E_i(u_\Gamma). \tag{6.6}$$

Since $(u - E_i(u_\Gamma))\,|_\Gamma = 0$, by Definition 6.1 we have:

$$a(u - E_i(u_\Gamma), E_i(u_\Gamma)) = 0. \tag{6.7}$$

Using (6.6) and (6.7), we have the energy minimizing property of $E_i(u_\Gamma)$ among $u \in L_{2,0}(\Omega^{(i)})$ with $u|_\Gamma = u_\Gamma$:

$$\begin{aligned} a(u,u) &= a(u - E_i(u_\Gamma), u - E_i(u_\Gamma)) + 2\,a(u - E_i(u_\Gamma), E_i(u_\Gamma)) + a(E_i(u_\Gamma), E_i(u_\Gamma)) \\ &\geq a(E_i(u_\Gamma), E_i(u_\Gamma)). \end{aligned} \tag{6.8}$$

Therefore, using (6.8), (6.4) and (4.18), we have:

$$s_i(u_\Gamma, u_\Gamma) \leq a(u,u) \leq \overline{\lambda}\ \delta^d\ \|u\|^2_{L_2(\Omega^{(i)})},$$

for all $u \in L_{2,0}(\Omega^{(i)})$, in particular, for $u\ \chi_\Gamma$. Hence,

$$s_i(u_\Gamma, u_\Gamma) \leq \overline{\lambda}\ \delta^d\ \|u\|^2_{L_2(\Gamma)}. \tag{6.9}$$

For the lower bound, we simply use (6.3) and (4.11):

$$\lambda_{Pncr,\,2}\,\delta^m \|u\|^2_{L_2(\Gamma)} \leq \lambda_{Pncr,\,2}\,\delta^m \|E_i(u_\Gamma)\|^2_{L_2(\Omega)} \leq a(E(u_\Gamma), E(u_\Gamma)) = s(u_\Gamma, u_\Gamma). \tag{6.10}$$

We have proved the following spectral equivalence result:

Theorem 6.1. *For any $q \in L_2(\Gamma)$, we have:*

$$\lambda_{Pncr,\,2}\ \delta^m \leq \frac{s_i(q,q)}{\|q\|^2_{L_2(\Gamma)}} \leq \overline{\lambda}\ \delta^d. \tag{6.11}$$

Thus, the condition number of the Schur complement matrix $S_\Gamma := S^{(1)} + S^{(2)}$ has the following bound:

$$\kappa(S_\Gamma) \lesssim \delta^{d-m}.$$

Remark 6.1. *The preceeding condition number estimate indicates that the condition number of the Schur complement is no greater than that of the corresponding stiffness matrix; see (4.21). This estimate is not tight. In fact, we numerically observe smaller condition numbers for the Schur complement; see Table 6.3.*

6.2.1. The Nonlocal Schur Complement Matrix. When the contributions from each subdomain are accounted separately, we can write $K_{\Gamma\Gamma}$ in (6.2) as $K_{\Gamma\Gamma} = K^{(1)}_{\Gamma\Gamma} + K^{(2)}_{\Gamma\Gamma}$. Then, $S^{(i)}$ in (6.5) can be written as follows:

$$S^{(i)} := K^{(i)}_{\Gamma\Gamma} - K_{\Gamma i} K^{-1}_{ii} K_{i\Gamma}.$$

The solution across the whole of Γ is determined by solving $S_\Gamma u_\Gamma = \tilde{f}$ for u_Γ, where

$$\tilde{f} := f_\Gamma - K_{\Gamma 1} K^{-1}_{11} f_1 - K_{\Gamma 2} K^{-1}_{22} f_2.$$

We proved in §4.4 and verified numerically in §4.5 that the condition number of the stiffness matrix K is h-independent. Therefore the condition number of the Schur complement matrix S_Γ should also be h-independent. We will examine this conjecture numerically in §6.3.

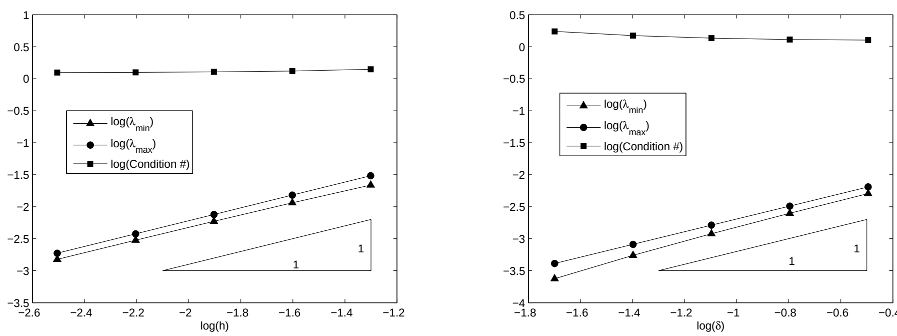
(a) Constant δ, vary h.

$1/h$	δ	$\lambda_{\min} \neq 0$	$\lambda_{\max}$	Condition #
20	0.3	2.18E-02	3.05E-02	1.40E+00
40	0.3	1.15E-02	1.52E-02	1.32E+00
80	0.3	5.91E-03	7.55E-03	1.28E+00
160	0.3	2.99E-03	3.77E-03	1.26E+00
320	0.3	1.51E-03	1.88E-03	1.25E+00

(b) Constant h, vary δ.

$1/h$	δ	$\lambda_{\min} \neq 0$	$\lambda_{\max}$	Condition #
100	0.02	2.37E-04	4.10E-04	1.73E+00
100	0.04	5.49E-04	8.17E-04	1.49E+00
100	0.08	1.20E-03	1.63E-03	1.36E+00
100	0.16	2.49E-03	3.23E-03	1.30E+00
100	0.32	5.07E-03	6.44E-03	1.27E+00

TABLE 6.1
Condition number for S_Γ in 1D for (a) constant δ, allowing h to vary, and (b) constant h, allowing δ to vary. This data is plotted in Figure 6.1.



(a) Constant δ, vary h. (b) Constant h, vary δ.

FIG. 6.1. *Condition number for S_Γ in 1D for (a) constant δ, allowing h to vary, and (b) constant h, allowing δ to vary. The condition number of S_Γ is apparently nearly independent of both h and δ in 1D. These figures are plotted from data in Table 6.1.*

6.3. Numerical Verification of the Schur Complement Conditioning. To test the conjecture of the previous section, we discretize the pure Neumann boundary value problem

$$s(u, u) = 0 \qquad x \in \Gamma, \tag{6.12}$$

where d is the spatial dimension, using piecewise constant shape functions on uniform cartesian mesh, and numerically determine the largest and smallest nonzero eigenvalues, defining the effective condition number of the problem.

6.3.1. Results in One Dimension. We define the regions $\Omega_1 = (0, 0.5 - \delta/2)$, $\Omega_2 = (0.5+\delta/2, 1)$, and $\Gamma = (0.5-\delta/2, 0.5+\delta/2)$, such that Γ is always a region of width δ centered at $x = 0.5$. We then compute the largest and smallest nonzero eigenvalues of S_Γ.

We first compute the condition number of S_Γ for different h while holding δ constant to verify our conjecture that the condition number of S_Γ is h-independent. Our results appear in Tables 6.1 and Figures 6.1. The minimum and maximum

(a) Constant δ, vary h.

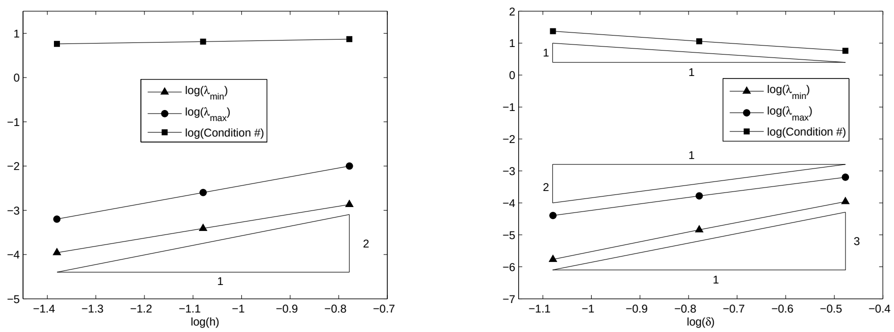
$1/h$	δ	$\lambda_{\min} \neq 0$	$\lambda_{\max}$	Condition #
6	0.333	1.34609E-03	9.96E-03	7.40E+00
12	0.333	3.88334E-04	2.52E-03	6.48E+00
24	0.333	1.09824E-04	6.32E-04	5.75E+00

(b) Constant h, vary δ.

$1/h$	δ	$\lambda_{\min} \neq 0$	$\lambda_{\max}$	Condition #
24	0.083	1.70E-06	4.01E-05	2.36E+01
24	0.167	1.44E-05	1.65E-04	1.14E+01
24	0.333	1.10E-04	6.32E-04	5.75E+00

Table 6.2

Condition number for S_Γ in 2D for (a) constant δ, allowing h to vary, and (b) constant h, allowing δ to vary. This data is plotted in Figure 6.2.



(a) Constant δ, vary h. (b) Constant h, vary δ.

Fig. 6.2. *Condition number for S_Γ in 2D for (a) constant δ, allowing h to vary, and (b) constant h, allowing δ to vary. The condition number of S_Γ in 2D is nearly independent of h but varies with δ^{-1}. These figures are plotted from data in Table 6.2.*

(a) 1D.

$1/h$	δ	Cond. # K	Cond. # S_Γ
20	0.30	9.41E+00	1.40E+00
40	0.30	8.92E+00	1.32E+00
80	0.30	8.58E+00	1.28E+00
160	0.30	8.39E+00	1.26E+00
320	0.30	8.29E+00	1.25E+00
100	0.02	1.55E+03	1.73E+00
100	0.04	4.49E+02	1.49E+00
100	0.08	1.16E+02	1.36E+00
100	0.16	2.93E+01	1.30E+00
100	0.32	7.51E+00	1.27E+00

(b) 2D.

$1/h$	δ	Cond. # K	Cond.# S_Γ
6	0.333	1.71E+01	7.40E+00
12	0.333	1.62E+01	6.48E+00
24	0.333	1.61E+01	5.75E+00
24	0.083	1.26E+02	2.36E+01
24	0.167	3.86E+01	1.14E+01
24	0.333	1.61E+01	5.75E+00

Table 6.3

Condition numbers of K and S_Γ in 1D and 2D. The condition number of the Schur complement is less than the condition number of the original system. This data is a summary of data from Tables 4.1, 4.2, 6.1, and 6.2.

eigenvalues depend linearly on h, with a slope of unity. Consequently, the condition number of S_Γ is h-independent. We then compute the condition number of S_Γ for different δ while holding h constant, and observe that the condition number is nearly δ independent.

The condition number of the Schur complement of the weak classical (local) Laplace operator varies as $(Hh)^{-1}$, where H is the subdomain size [40, Lemma 4.1.1]. Assuming δ "plays the role" of h as in §4.5.1, one might expect that the condition number of the Schur complement for the weak nonlocal Laplace operator to vary as $(H\delta)^{-1}$, but Figure 6.1(b) shows that this is clearly not the case in 1D.

6.3.2. Results in Two Dimensions. We define the regions $\Omega_1 = (0, 0.5 - \delta/2) \times (0,1)$, $\Omega_2 = (0.5+\delta/2, 1) \times (0,1)$, and $\Gamma = (0.5-\delta/2, 0.5+\delta/2) \times (0,1)$, such that Γ is always a region of width δ centered at $x = 0.5$. We then compute the largest and smallest nonzero eigenvalues of S_Γ.

We first compute the condition number of S_Γ for different h while holding δ constant to verify our conjecture that the condition number of S_Γ is h-independent. Our results appear in Tables 6.2 and Figures 6.2. The minimum and maximum eigenvalues depend linearly on h, with a slope of approximately two. Consequently, the condition number of S_Γ is nearly h-independent. We then compute the condition number of S_Γ for different δ while holding h constant, and observe that the condition number varies as δ^{-1}.

Dim	$\lambda_{\min}^{-1}(K)$	$\lambda_{\max}(K)$	$\kappa(K)$	$\lambda_{\min}^{-1}(S_\Gamma)$	$\lambda_{\max}(S_\Gamma)$	$\kappa(S_\Gamma)$
1D	$\mathcal{O}(\delta^{-3})$	$\mathcal{O}(\delta)$	$\mathcal{O}(\delta^{-2})$	$\mathcal{O}(\delta^{-1})$	$\mathcal{O}(\delta)$	$\mathcal{O}(1)$
2D	$\mathcal{O}\left(\delta^{-3.5}\right)$	$\mathcal{O}(\delta^2)$	$\mathcal{O}(\delta^{-1.5})$	$\mathcal{O}(\delta^{-3})$	$\mathcal{O}(\delta^2)$	$\mathcal{O}(\delta^{-1})$

Table 7.1
The δ-quantification of the reported numerical results.

7. Conclusions and Future Work. We collect the numerical results in Table 7.1. We observe that $\kappa(K)$ is independent of the mesh size, and that and $\kappa(S_\Gamma)$ is nearly so. The numerical estimates are computed using a piecewise constant discretization. Therefore, in order to show the sharpness of the estimates analytically, one can use the same piecewise constant functions. This is a subject for future work.

We conclude the sharpness of the upper bound $\mathcal{O}(\delta^d)$ in the spectral equivalence for $a(u,u)$ and $s(u,u)$, i.e., (4.20) and (6.11), respectively. When the condition numbers of the nonlocal and local problems are compared, we see that $\kappa(K) = \mathcal{O}(\delta^{-2})$ is analogous to the local case, where the condition number varies as $\mathcal{O}(h^{-2})$. However, this does not hold for higher dimensions because $\kappa(K) = \mathcal{O}(\delta^{d-m})$, and hence, depends on the spatial dimension unlike in the local case. By demonstrating the nonlocal problem converges to the local one as $\delta \to 0$, we showed that $m = 3$ is sharp in 1D. Numerically we observed that $m \approx 3.5$ in 2D. Also, the fractional power is somewhat surprising. A systematic and constructive mechanism for obtaining the values of $m = m(d)$ in higher dimensions is unclear. Therefore, more comprehensive study is needed to explore the dependence of m upon, for instance, the underlying basis functions and spatial dimension. Such study is also needed for the Schur complement to obtain sharp bounds. The effect of subdomain sizes on the conditioning of the Schur complement is unknown.

Application of an appropriate preconditioner, involving the solution of a coarse problem, reduces the condition number of the Schur complement of the weak classical (local) Laplace operator from $\mathcal{O}\left((Hh)^{-1}\right)$ to $\mathcal{O}\left((1+\log(H/h))^2\right)$ [40, Lemma 4.11], [39, §4.3.6]. One unexplored area involves examining the role of a coarse problem in the nonlocal setting, which has not been considered here. A logical direction would be to expand other substructuring methods to a nonlocal setting, such as Neumann-Dirichlet, Neumann-Neumann, FETI-DP (the dual-primal finite element tearing and interconnecting method) [26], or BDDC (balancing domain decomposition by constraints) [15]. Additional opportunities for future research include addressing convergence analysis for alternative domain decomposition methods not based on substructuring in a nonlocal setting. More fundamental concepts in Schwarz theory such as stable decompositions and local solvers need to be reconstructed for nonlocal problems to support convergence analysis for additive, multiplicative, and hybrid algorithms.

Appendix A. The Poincaré constant is independent of $\boldsymbol{\delta}$.

In order to show that the Poincaré constant λ_{Pncr} is independent of δ, we give the construction in detail that leads to (4.6). Let the radius of inscribing and circumscribing spheres of Ω be denoted r_{in} and r_{out}, respectively. We compare the volume of B_j to the volume of the annuli of width $\frac{\delta^m}{2}$ of the inscribing and circumscribing spheres.

In 2D, we give the explicit expression of the area of circumscribing annulus and, for sufficiently small δ, we have the following:

$$
\begin{aligned}
|B_{j,out}| &= \pi \left[(r_{out} - \{j-1\}\frac{\delta^m}{2})^2 - (r_{out} - j\,\frac{\delta^m}{2})^2 \right] \\
&= \pi\,\frac{\delta^m}{2}\left[2\,r_{out} - (j-\frac{1}{2})\,\delta^m\right] \\
&\approxeq r_{out}\,\delta^m.
\end{aligned}
$$

In 3D, the volume of the inscribing annulus is given by the following:

$$
\begin{aligned}
|B_{j,out}| &= \frac{4}{3}\pi \left[(r_{out} - \{j-1\}\frac{\delta^m}{2})^3 - (r_{out} - j\,\frac{\delta^m}{2})^3 \right] \\
&= \frac{4}{3}\pi\,\frac{\delta^m}{2}\left[(r_{out} - \{j-1\}\frac{\delta^m}{2})^2 + (r_{out} - \{j-1\}\frac{\delta^m}{2})(r_{out} - j\,\frac{\delta^m}{2}) + (r_{out} - j\,\frac{\delta^m}{2})^2 \right] \\
&\approxeq r_{out}^2\,\delta^m.
\end{aligned}
$$

Similarly, $|B_{j,in}| \approxeq r_{in}^{d-1}\,\delta^m$. Hence,

$$
\underline{c_{in}}\,r_{in}^{d-1}\,\delta^m \le |B_{j,in}| \le |B_j| \le |B_{j,out}| \le \overline{c_{out}}\,r_{out}^{d-1}\,\delta^m. \tag{A.1}
$$

Note that both $\underline{c_{in}}\,r_{in}^{d-1}$ and $\overline{c_{out}}\,r_{out}^{d-1}$ depend only on Ω. Consequently, in any spatial dimension, we conclude (4.6), i.e., $c_{in}\,\delta^m \le |B_j| \le c_{out}\,\delta^m$, where c_{in} and c_{out} depend only on Ω.

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